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Effect of A Very Strong Magnetic Field on Vorticity Covariance in MHD Turbulence

Abstract

In this paper we have discussed the effect of a strong magnetic field on vorticity covariance. The rate of change of vorticity covariance is derived in terms of defining scalars α(r, t), β(r, t) and γ(r, t).

Keywords: Vorticity covariance, Isotropic Turbulence, Magnetic field, Fluid dynamic Turbulence.

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Introduction

The statistical theory of homogeneous isotropic turbulence initiated by G.I. Taylor (1935) extended by Van Karman and L. Howarth (1948) and verified and criticized by H.P. Robertson (1940) and S. Chandrasekhar (1959) has much value in the problems of fluid dynamics. A good deal of theoretical studies on magneto-dynamic turbulence have been made in these several years. Some authors considered MHD turbulence in the absence of an external magnetic field in order to gain a basic understanding of a self-adjusting processes of the mechanical and magnetic modes of turbulence. Particularly S. Chandrasekhar (1959) extended his space-time correlation method as well as Heisenberg's picture of and "eddy viscosity" to the MHD case, while T. Tatsumi (1960) worked out with the so-called zero fourth cumulant hypothesis. In a variety of astro and geophysical problems, however, it is often the case that a certain external magnetic field such as the cosmic magnetic field, the geometric field, etc; is imposed on a turbulent motion of a conducting fluid. E. Lehnert (1955) discussed the final stage of decaying MHD turbulence in a uniform magnetic field and contribution to this problem were also made by G.S. Golitsyn (1960) and H. Moffat (1961). The essential effect of the presence of an imposed magnetic field is that the mechanical and magnetic modes of turbulence interact not only with each other through the self-adjusting processes but also with the external magnetic field. If the external field is very strong, the effect of the latter interaction will predominate over the self-adjusting process. Michio Ohji (1964) presented a first order theory for turbulence of an electrically conducting fluid in the presence of a uniform magnetic field which so strong that the non-linear mechanism as well as the dissipation terms are of minor importance compared to the external coupling terms. In his other paper, Ohji (1978), discussed the effect of uniform magnetic field on incompressible – MHD turbulence in the presence of a constant angular velocity and Hall effect. In this paper, we have discussed the effect of a uniform magnetic field on vorticity covariance in MHD turbulence. The expression for vorticity covariance is obtained in terms of defining scalars as given in Mishra and Kishore (1970). It is assumed that the imposed field is strong enough for non-linear coupling terms while the reactions from the turbulence are assumed to be negligible.

Aim of the Study

In this paper, we have discussed the effect of a uniform magnetic field on vorticity covariance in MHD turbulence. The expression for vorticity covariance is obtained in terms of defining scalars as given in Mishra and Kishore (1970). It is assumed that the imposed field is strong enough for non-linear coupling terms while the reactions from the turbulence are assumed to be negligible.

1. Basic Equation

If $\overline{U}$ denotes the velocity, $\overline{B}$ the magnetic induction and P the pressure, the MHD equations for a conductivity fluid of the density ρ, the kinematic viscosity v_r the conducting σ and the permeability μ are written, in MKS units (Ohji, 1978)

$$\frac{\partial \bar{U}}{\partial t}+\left(\bar{U}\ \text{grad}\right)\bar{U}-\frac{1}{\rho\mu}\left(\bar{B}\ \text{grad}\right)\bar{B}=-\frac{1}{\rho}\text{grad}\left(P+\frac{1}{2\mu}\bar{B}^2\right)+\nu\nabla^2\bar{U}. \tag{1.1}$$

for the momentum, and

$$\frac{\partial \bar{B}}{\partial t}+\left(\bar{U}\ \text{grade}\right)\bar{B}-\left(\bar{B}\ \text{grad}\right)\bar{U}=\frac{1}{\mu\sigma}\nabla^2\bar{B} \tag{1.2}$$

for the induction, respectively, together with the supplementary equations

$$\text{div}\ \bar{U}=0 \quad \text{and} \quad \bar{B}=0, \tag{1.3}$$

where ρ, v, σ and μ are constants.

Further, it is convenient to introduce the Alfven velocity

$$\bar{H}=\bar{B}/\sqrt{(\mu\rho)} \tag{1.4}$$

and the magnetic viscosity

$$\lambda=\frac{1}{\mu\sigma} \tag{1.5}$$

For a turbulent flow, we put

$$\underline{W}=\underline{\bar{U}}+\underline{u},\ \underline{H}=\bar{H}+h,\ P=\bar{P}+p, \tag{1.6}$$

where $\bar{U},\bar{H}$ and $\bar{P}$ are the mean values of u, h and p represent the respective fluctuating components. Then, taking the statistical average (expressed by an over bar) of equations (1.1) (1.2), we have, in the usual index notations:

$$\frac{\partial \bar{U}_i}{\partial t}+\frac{\partial}{\partial x_k}\left[\bar{U}_i\bar{U}_k-\bar{H}_i\bar{H}_k+\bar{u}_i\bar{u}_k-\bar{h}_i\bar{h}_k\right]$$

$$=-\frac{1}{\rho}\frac{\partial}{\partial x_i}\left[P+\frac{1}{2}\left(\bar{H}^2+h^2\right)\right]+\nu\Delta^2\bar{U}_i \tag{1.7}$$

$$\frac{\partial \bar{H}_i}{\partial t}+\frac{\partial}{\partial x_k}\left[\bar{H}_i\bar{U}_k-\bar{U}_i\bar{H}_k+\bar{h}_i\bar{u}_k-\bar{u}_i\bar{h}_k\right]=\lambda\Delta^2\bar{H}_i \tag{1.8}$$

$$\frac{\partial \bar{U}_i}{\partial x_i}+\frac{\partial \bar{H}_i}{\partial x_k}=0 \tag{1.9}$$

for the mean fields, and substracting these from equations (1.1), (1.3), we obtain

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_k}\left(\bar{u}_i\bar{u}_k-\bar{h}_i\bar{h}_k\right)+\bar{U}_k\frac{\partial u_i}{\partial x_k}-\bar{H}_k\frac{\partial h_i}{\partial x_k}$$

$$=-\frac{1}{\rho}\frac{\partial}{\partial x_i}\left[P+\frac{1}{2}\left(h^2+2\bar{H}_k h_k\right)\right]$$

$$+\nu\nabla^2 u_i+\left\{\frac{\partial}{\partial x_k}\left(u_i u_k-h_i h_k\right)-u_k\frac{\partial \bar{U}_i}{\partial x_k}+h_k\frac{\partial \bar{H}_i}{\partial x_k}+\frac{1}{2}\frac{\partial h^2}{\partial x_i}\right\} \tag{1.10}$$

$$\frac{\partial H_i}{\partial t}+\frac{\partial}{\partial_i}\left(h_i u_k-u_i h_k\right)\bar{U}_k\frac{\partial h_i}{\partial x_k}-\bar{H}_k\frac{\partial u_i}{\partial x_k}$$

$$=\lambda\ \nabla^2 h_i+\left\{\frac{\partial}{\partial x_i}\left(h_i u_k-u_i h_k\right)-u_k\frac{\partial \bar{H}}{\partial x_k}+h_k\frac{\partial \bar{U}_i}{\partial x_k}\right\} \tag{1.11}$$

and

$$\frac{\partial u_i}{\partial x_i}=\frac{\partial h_i}{\partial x_i}=0, \tag{1.12}$$

for the fluctuating field. Especially, if both $\bar{U}$ and $\bar{H}$ are steady, uniform and moreover the turbulence is homogeneous, the averaged equations (1.7), (1.8) are satisfied identically and it is readily seen that in equations (1.10) and (1.11), the terms in the curly brackets vanish.

It is assumed that the imposed field is strong enough for non-linear coupling terms while reactions from the turbulence are considered to be negligible. In this case, the equations (1.10), (1.11) becomes

$$\frac{\partial u_i}{\partial t}-\bar{H}_k\frac{\partial h}{\partial x_k}=-\frac{\partial w}{\partial x_i}+\nu\nabla^2 u_i, \tag{1.13}$$

$$\frac{\partial h_i}{\partial t}-\bar{H}_k\frac{\partial u}{\partial x_k}=\lambda\nabla^2 h_i, \tag{1.14}$$

where

$$w=\frac{P}{\rho}+\bar{H}_k h_k$$

2. Solution of the Problem

Taking curl of (1.13)

$$\frac{\partial \omega_i}{\partial t}-H_k\frac{\partial J_i}{\partial x_k}-M_k\frac{\partial h_i}{\partial x_k}=\nu\frac{\partial^2\omega_i}{\partial x_k\partial x_k} \tag{2.1}$$

where $J_i=\nabla\times h_i,\quad M_k=\nabla\times H_k$

$$\frac{\partial \omega_j^{'}}{\partial t}-H_k^{'}\frac{\partial J_j^{'}}{\partial x_k}-M_j^{'}\frac{\partial h_j^{'}}{\partial x_k^{'}}=\nu\frac{\partial^2\omega_j^{'}}{\partial x_k^{'}\partial x_k^{'}} \tag{2.2}$$

Multiplying equation (2.1) by $\omega_j^{'}$ and (2.2) by ω_i we get

$$\omega_j^{'}\frac{\partial \omega_i}{\partial t}-\omega_j^{'}H_k\frac{\partial J_i}{\partial x_k}-\omega_j^{'}M_k\frac{\partial h_i}{\partial x_k}=\nu\omega_j^{'}\frac{\partial^2\omega_i}{\partial x_k\partial x_k} \tag{2.3}$$

and

$$\omega_i \frac{\partial \omega_j'}{\partial t} - \omega_i H_k' \frac{\partial J_j'}{\partial x_k} - \omega_i M_k \frac{\partial h_j'}{\partial x_k'} = \nu \omega_i \frac{\partial^2 \omega_j'}{\partial x_k' \partial x_k'} \quad (2.4)$$

Adding equations (2.3) and (2.4) and taking ensemble average, we have

$$\frac{\partial \overline{\omega_i \omega_j'}}{\partial t} - \frac{\partial \overline{J_j \omega_j' H_k}}{\partial x_k} - \frac{\partial \overline{J_j' \omega_i H_k'}}{\partial x_k'} - \frac{\partial \overline{h_j' \omega_i M_k'}}{\partial x_k'} = \nu \omega_j' \frac{\partial^2 \omega_i}{\partial x_k \partial x_k} + \nu \omega_i \frac{\partial^2 \omega_j'}{\partial x_k' \partial x_k'} \quad (2.5)$$

Now using $\xi_k = x_k' - x_k$ and $\frac{\partial}{\partial \xi_k} = -\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_k'}$ as a consequence of homogeneity, we have

$$\frac{\partial \overline{\omega_i \omega_j'}}{\partial t} + \frac{\partial \overline{J_j \omega_j' H_k}}{\partial \xi_k} - \frac{\partial \overline{J_j' \omega_i H_k'}}{\partial \xi_k} + \frac{\partial \overline{h_j \omega_i M_k}}{\partial \xi_k} - \frac{\partial \overline{h_j' \omega_i M_k'}}{\partial \xi_k} = 2\nu \frac{\partial^2 \omega_i \omega_j'}{\partial \xi_k \partial \xi_k} \quad (2.6)$$

let

$$\overline{H_k J_i \omega_j'} = S_{ki,j}$$
$$\overline{H_k' J_j' \omega_j} = R_{kj,i}$$
$$\overline{h_i M_k \omega_j} = L_{ik,j}$$
$$\overline{\omega_i h_j' M_k'} = T_i, j_k$$

$$\frac{\partial \overline{\omega_i \omega_j'}}{\partial t} + \frac{\partial}{\partial \xi_k} S_{ki,j} - \frac{\partial}{\partial \xi_k} R_{kj,i} + \frac{\partial}{\partial x_k} L_{ik,j} - \frac{\partial}{\partial \xi_k} T_{i,jk} = 2\nu \frac{\partial^2 \omega_i \omega_j'}{\partial \xi_k \xi_k} \quad (2.7)$$

we have

$$\frac{\partial}{\partial \xi_k} S_{ki,j} = \left(S'' + \frac{4S'}{r} \right) \in_{ijl} \xi_l$$

and

$$\frac{\partial}{\partial \xi_k} R_{kj,i} = \left(R'' + \frac{4R'}{r} \right) \in_{kj\eta} \xi_\eta$$

$$\frac{\partial}{\partial \xi_k} L_{ik,j} = \left(L'' + \frac{4L'}{r} \right) \in_{ijm} \xi_m$$

$$\frac{\partial}{\partial \xi_k} T_{i,jk} = \left(T'' + \frac{4T'}{r} \right) \in_{ijq} \xi_q$$

$$\frac{\partial \overline{\omega_i \omega_j'}}{\partial t} + \left(S'' + \frac{4S'}{r} \right) \in_{ijl} \xi_l - \left(R'' + \frac{4R'}{r} \right) \in_{kj\eta} \xi_\eta + \left(L'' + \frac{4L'}{r} \right) \in_{ij,m} \xi_m - \left(T'' + \frac{4T'}{r} \right) \in_{ij,q} \xi_q = 2\nu \frac{\partial \omega_i \omega_j'}{\partial \xi_k \partial \xi_k} \quad (2.8)$$

let

$$\overline{\omega_i \omega_j'} = \alpha(r,t) \xi_i \xi_j + \beta(r,t) \delta_{ij} + \gamma(r,t) \in_{ijl} \xi_l \quad (2.9)$$

and

$$\frac{\partial^2 Q_{ij}}{\partial \xi_k \partial \xi_k} = \left(\frac{Q'''}{r} + \frac{4Q''}{r^2} - \frac{4Q'}{r^3} \right) \xi_i \xi_j - \left(rQ''' + 6Q'' + \frac{4Q'}{r} \right) \delta_{ij} \quad (2.10)$$

Therefore

$$\frac{\partial \alpha(r,t)}{\partial t} \xi_i \xi_j + \frac{\partial \beta(r,t)}{\partial t} \delta_{ij} + \frac{\partial \gamma(r,t)}{\partial t} \in_{ijl} \xi_l = -\left(S'' + \frac{4S'}{r} \right) \in_{ijl} \xi_l + \left(R'' + \frac{4R'}{r} \right) \in_{kj\eta} \xi_\eta - \left(L'' + \frac{4L'}{r} \right) \in_{ij,m} \xi_m - \left(T'' + \frac{4T'}{r} \right) \in_{ij,q} \xi_q + 2\nu \left(\frac{Q'''}{r} + \frac{4Q''}{r^2} - \frac{4Q'}{r^3} \right) \xi_i \xi_j - 2n \left(rQ''' + 6Q'' + \frac{4Q'}{r} \right) \delta_{ij} \quad (2.11)$$

Therefore

$$\frac{\partial \alpha}{\partial t} = 2\nu \left(\frac{Q'''}{r} + \frac{4Q''}{r^2} - \frac{4Q'}{r^3} \right) \quad (2.12)$$

$$\frac{\partial \beta}{\partial t} = -2\nu \left(rQ''' + 6Q'' + \frac{4Q'}{r} \right) \quad (2.13)$$

$$\frac{\partial \gamma}{\partial t} = -\left(S'' + \frac{4S'}{r} \right) + \left(R'' + \frac{4R'}{r} \right) - \left(L'' + \frac{4L'}{r} \right) - \left(T'' + \frac{4T'}{r} \right) \quad (2.14)$$

Conclusion

In the presence of strong magnetic field, we have derived the rate of decay of vorticity covariance in terms of defining scalars. The analytically expressions are given by equations (2.12), (2.13) and (2.14). In absence of imposed magnetic field these expressions reduce to

$$\frac{\partial \alpha}{\partial t} = 2\nu\left(\frac{Q'''}{r} + \frac{4Q''}{r^2} - \frac{4Q'}{r^3}\right) \tag{2.15}$$

$$\frac{\partial \beta}{\partial t} = -2\nu\left(rQ''' + 6Q'' + \frac{4Q'}{r}\right) \tag{2.16}$$

and

$$\frac{\partial \gamma}{\partial t} = 0 \tag{2.17}$$

which are the same given by Anupama et al. (2007). A point of future interest may perhaps be to elucidate the effect of the non-steady or non-uniform external fields as well as that of the non-linear energy transfer mechanism.

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